

Dyson-Schwinger equations II

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A recursion for γ_k

Write $G(x, L) = 1 - \sum \gamma_k L^k$

Looking for a recursion for γ_k in terms
of lower γ_j

Essentially this is the RGE.

→ derive it directly for RG
or
→ use $S * Y$

The renormalization group equation

For a vertex v

$$\left(\frac{\partial}{\partial L} + \beta(x) \frac{\partial}{\partial x} - \sum_{e \text{ adjacent to } v} \gamma^e(x) \right) x^{\text{val}(v)-2)/2} G^v(x, L) = 0$$

For an edge e

$$\left(\frac{\partial}{\partial L} + \beta(x) \frac{\partial}{\partial x} - 2\gamma^e(x) \right) G^e(x, L) = 0$$

where

$$\beta(x) = \partial_L x \phi_R(Q)|_{L=0} = \sum_{j \in \mathcal{R}} |s_j| \gamma_1^j(x) x$$

$$\gamma^e(x) = -\frac{1}{2} \partial_L G^e(x, L)|_{L=0} = \frac{1}{2} \gamma_1^e$$

Expanding the vertex

$$\begin{aligned}
 & \left(\frac{\partial}{\partial L} + \beta(x) \frac{\partial}{\partial x} - \sum_{e \sim v} \gamma^e(x) \right) x^{\frac{\text{val}(v)-2}{2}} G^v(x, L) = 0 \\
 &= x^{\frac{\text{val}(v)-2}{2}} \frac{\partial}{\partial L} G^v(x, L) + \beta(x) x^{\frac{\text{val}(v)-2}{2}} \frac{\partial}{\partial x} G^v(x, L) \\
 &\quad + x^{\frac{\text{val}(v)-2}{2}} \beta(x) \frac{\partial}{\partial x} G^v(x, L) - x^{\frac{\text{val}(v)-2}{2}} \frac{1}{2} \sum_{e \sim v} \gamma^e_1(x) G^v(x, L) \\
 &= x^{\frac{\text{val}(v)-2}{2}} \left(\frac{\partial}{\partial L} G^v(x, L) + \beta(x) \frac{\partial}{\partial x} G^v(x, L) + \left[\frac{\cancel{\text{val}(v)-2}}{2} \frac{1}{\cancel{\text{val}(v)-2}} \left(\gamma^v_1 + \sum_{e \sim v} \cancel{\gamma^e_1} \right) \right. \right. \\
 &\quad \left. \left. - \frac{1}{2} \sum_{e \sim v} \cancel{\gamma^e} \right] G^v(x, L) \right) \\
 &= x^{\frac{\text{val}(v)-2}{2}} \left(\frac{\partial}{\partial L} G^v(x, L) + \beta(x) \frac{\partial}{\partial x} G^v(x, L) + \gamma^v_1 G^v(x, L) \right)
 \end{aligned}$$

$$0 = \left(\frac{\partial}{\partial L} + \beta(x) \frac{\partial}{\partial x} + \gamma_1^v(x) \right) G^v(x, L)$$

$$0 = \left(\frac{\partial}{\partial L} + \beta(x) \frac{\partial}{\partial x} - \gamma_1^e(x) \right) G^e(x, L)$$

$$\left(\frac{\partial}{\partial L} + \sum_{j \in \mathcal{R}} |s_j| \gamma_1^j(x) x \frac{\partial}{\partial x} + \text{sign}(s_r) \gamma_1^r(x) \right) G^r(x, L) = 0.$$

The γ_k recursion

Extracting the coefficient of L^{k-1} and rearranging gives

$$G^r(x, L) = 1 \pm \sum_k \gamma_k^r L^k$$

$$k\gamma_k^r + \sum_j |s_j| \gamma_1^j(x) \times \frac{\partial}{\partial x} \gamma_{k-1}^r + \text{sgn}(s_r) \gamma_1^r(x) \gamma_{k-1}^r = 0$$

$$\text{so } \gamma_k^r = \frac{1}{k} \left(\gamma_1^r(x) \text{sgn}(s_r) - \sum_j |s_j| \gamma_1^j(x) \times \partial_x \right) \gamma_{k-1}^r$$

Specializing to the single equation case gives

$$\gamma_k = \frac{1}{k} \gamma_1(x) (\text{sign}(s) - |s| x \partial_x) \gamma_{k-1}(x).$$

$S \star Y$ – some definitions

Some definitions

- Let S be the antipode of the Hopf algebra.

- Let Y be the grading operator. $Y(-\text{cup}) = 3 - \text{cup}$

- Let

$$\sigma_1 = \partial_L \phi_r(S \star Y)|_{L=0}$$

and

$$\sigma_n = \frac{1}{n!} m^{n-1} (\underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{n \text{ times}}) \Delta^{n-1}$$

$S \star Y$ – some lemmas

- $S \star Y$ is 0 on products

-

$$\Delta([x^k]X^r) = \sum_{j=0}^k [x^j]X^r Q^{k-j} \otimes [x^{k-j}]X^r$$

$$\Delta([x^k]X^r Q^\ell) = \sum_{j=0}^k [x^j]X^r Q^{k+\ell-j} \otimes [x^{k-j}]X^r Q^\ell$$

Walter's
formula
in a
different

-

$$(P_{\text{lin}} \otimes \text{id})\Delta X^r = X^r \otimes X^r - \sum_{j \in \mathcal{R}} s_j X^j \otimes x \partial_x X^r$$

pulling out
coeff $\rightarrow$ like a derivative.

$$(P_{\text{lin}} \otimes \text{id})\Delta X = X \otimes X - sX \otimes x \partial_x X$$

for the single equation case

The scattering type formula

The scattering type formula captures the renormalization group in a Connes-Kreimer framework. In our notation it says

$$\sigma_n(X^r) = \text{sign}(s) \gamma_n^r(x)$$

$$\left(\begin{array}{l} \text{the DSE is} \\ X = \underline{1} \pm \sum_k X^k B_+^k (X Q^k) \\ \text{in single cqn case} \end{array} \right)$$

The γ_k recursion

Using

$$\sigma_n(X) = \text{sign}(s)\gamma_n$$

$$\sigma_1 = \partial_L \phi_r(S \star Y)|_{L=0}$$

$$\sigma_n = \frac{1}{n!} m^{n-1} \underbrace{(\sigma_1 \otimes \cdots \otimes \sigma_1)}_{n \text{ times}} \Delta^{n-1}$$

$$(P_{\text{lin}} \otimes \text{id})\Delta X = X \otimes X - sX \otimes x\partial_x X$$

Calculate

$$\gamma_n = \text{sign}(s) \sigma_n(X) = \text{sign}(s) \frac{1}{n!} m^{n-1} (\sigma_1 \otimes \cdots \otimes \sigma_1) \Delta^{n-1} X$$

$$= \text{sign}(s) \frac{1}{n} m (\sigma_1 \otimes \sigma_{n-1}) \Delta X$$

$$= \text{sign}(s) \frac{1}{n} m (\sigma_1 \otimes \sigma_{n-1}) (P_{\text{lin}} \otimes \text{id}) \Delta X$$

$$\begin{aligned}
&= \text{sgn}(s) \frac{1}{n} m(\sigma_1 \otimes \sigma_{n-1}) (X \otimes X - s X \otimes x \partial_x X) \\
&= \frac{1}{n} (\gamma_1 \gamma_{n-1} - s \gamma_1 x \partial_x \gamma_{n-1})
\end{aligned}$$

$$\gamma_n = \frac{1}{n} \gamma_1 (1 - s x \partial_x) \gamma_{n-1}$$

And similarly for systems.

Notes

1. As series in x , γ_k has lowest term x^k
2. Because we're renormalizing by subtraction there's
no γ_0 term

Trading ρ for x

$$\sum \gamma_k \partial_{-\rho}^k$$

Let $D = \text{sign}(s) \gamma \cdot \partial_{-\rho}$ and $F_k(\rho) = \sum_{i=0}^{t_k} F_{k,i}(\rho)$ so the Dyson-Schwinger equation reads

$$\gamma \cdot L = \sum_{k \geq 1} x^k (1 - D)^{1-sk} (e^{-L\rho} - 1) F_k(\rho) \Big|_{\rho=0}$$

What is the lowest possible degree of x in

$$x^k (1 - D)^{1-sk} \rho^\ell \Big|_{\rho=0}$$

$$k + \left(\text{lowest degree of } x \text{ in } \gamma_{k_1} \dots \gamma_{k_n} \right)$$

where $k_1 + \dots + k_n = \ell$

$$= k + \ell .$$

Reduction to geometric series

So, there exists unique $r_k, r_{k,i} \in \mathbb{R}$, $k \geq 1$, $1 \leq i < k$ such that

$$\sum_k x^k (1-D)^{1-sk} (e^{-L\rho} - 1) \underbrace{F_k(\rho)}_{\text{good part}} \Big|_{\rho=0}$$

$$= \sum_k x^k (1-D)^{1-sk} (e^{-L\rho} - 1) \left(\underbrace{\frac{r_k}{\rho(1-\rho)}}_{\text{good part}} + \underbrace{\sum_{1 \leq i < k} \frac{r_{k,i} L^i}{\rho}}_{\text{hack part}} \right) \Big|_{\rho=0}$$

Note

Mysterious series

The $r_{i,j}$ are mysterious. With $s = 2$ and a single primitive at 1 loop we get.

with
$$F(p) = \frac{f_{-1}}{p} + f_0 + f_1 p + \dots$$

$$r_1 = f_{-1}$$

$$r_2 = f_{-1}^2 - f_{-1}f_0$$

$$r_{2,1} = 0$$

$$r_3 = 2f_{-1}^3 + f_{-1}^2(-4f_0 + f_1) + f_{-1}f_0^2$$

$$r_{3,1} = -f_{-1}^3 + f_{-1}^2f_0$$

$$r_{3,2} = 0$$

$$r_4 = 2f_{-1}^4 + f_{-1}^3(-12f_0 + 6f_1 - f_2) + f_{-1}^2(9f_0^2 - 3f_0f_1) - f_{-1}f_0^3$$

$$r_{4,1} = -f_{-1}^4 + f_{-1}^3(6f_0 - 2f_1) - 3f_{-1}^2f_0^2$$

$$r_{4,2} = \frac{7}{6}f_{-1}^4 - \frac{7}{6}f_{-1}^3f_0$$

$$r_{4,3} = 0$$

$\vdots$

Even particular coefficients and examples are mysterious.

$$F(\rho) = \frac{-1}{\rho(1-\rho)(2-\rho)(3-\rho)},$$

gives

$r_1 = -\frac{1}{6}$		
$r_2 = -\frac{5}{6^3}$	$r_{2,1} = 0$	
$r_3 = -\frac{14}{6^5}$	$r_{3,1} = \frac{-5}{6^4}$	$r_{3,2} = 0$
$r_4 = \frac{563}{6^7}$	$r_{4,1} = \frac{-173}{6^6}$	$r_{4,2} = \frac{-35}{6^6}$
$r_5 = \frac{13030}{6^9}$	$\vdots$	$\vdots$
$r_6 = -\frac{194178}{6^{11}}$		

L and L^2

The Dyson-Schwinger equation has become

$$\gamma \cdot L = \sum_k x^k (1-D)^{1-sk} (e^{-\cancel{L}\rho} - 1) \left(\frac{r_k}{\rho(1-\rho)} + \sum_{1 \leq i < k} \frac{r_{k,i} L^i}{\rho} \right) \Big|_{\rho=0}$$

Take the coefficients of L and L^2

$$[L]: \quad \gamma_1 = - \sum_k x^k (1-D)^{1-sk} \cancel{\rho} \frac{\cancel{\rho} r_k}{\cancel{\rho}(1-\rho)} \Big|_{\rho=0}$$

$$[L^2]: \quad 2\gamma_2 = \sum_k x^k (1-D)^{1-sk} \left(\frac{\rho r_k}{(1-\rho)} - 2r_{k,1} \right) \Big|_{\rho=0}$$

$$= - \sum_k x^k (1-D)^{1-sk} \left(r_k - 2r_{k,1} \right) \Big|_{\rho=0}$$

$$6-1 + \sum_k x^k (1-D)^{1-sk} \left(\frac{r_k}{1-\rho} \right) \Big|_{\rho=0}$$

$$\begin{aligned}
2\gamma_2 &= - \sum_k x^k (1-D)^{1-sk} (r_k + 2r_{k,1})|_{p=0} - \gamma_1 \\
&= - \sum_k x^k (r_k + 2r_{k,1})|_{p=0} - \gamma_1
\end{aligned}$$

$$2\gamma_2 = -\gamma_1 - \sum_{k \geq 1} (r_k + 2r_{k,1})x^k$$

The γ_1 recursion

Recall

$$\gamma_k = \frac{1}{k} \gamma_1(x) (\text{sign}(s) - |s| x \partial_x) \gamma_{k-1}(x).$$

Let

$$P(x) = - \sum_{k \geq 1} (r_k + 2r_{k,1}) x^k$$

Get

$$\gamma_1(x) (\text{sign}(s) - |s| x \partial_x) \gamma_1(x) = P(x) - \gamma_1(x)$$

Philosophy

- In defining the $r_k, r_{k,i}$ I made it a geometric series in L^0 but $\frac{1}{p}$ for higher L
- If I used $\frac{1}{p}$ in all cases then one L deriv gives $\delta_i(x) = \sum r_k x^k$ so no recursion
- If geometric series for all $r_{k,i}$ then the ∂_p don't straightforwardly disappear
- Geometric series keeps the conformal invariance

The goal is

Balance

- giving results
- putting too much info in P
- representing the underlying physics.

How to interpret $P(x)$

In the original Yukawa example of Dirk and David

$$P(x) = x \quad - \text{no extra garbage.}$$

→ reducing to one insertion place → already the case

→ $F(p)$ already a geometric series.

The reduction to one insertion place
is just a case of building new primitives

The reduction to geometric series just rearranges coefficients
still think of P as the function for the primitives

Systems

The shape of the final equations in the system case is

$$\gamma_1^r = \sum_{k \geq 1} p^r(k) x^k - \text{sign}(s_r) \gamma_1^r(x)^2 + \sum_{j \in \mathcal{R}} |s_j| \gamma_1^j(x) x \partial_x \gamma_1^r(x)$$

Summary of the big picture

The Broadhurst Kreimer example

$$G(x, L) = 1 - \frac{x}{q^2} \int d^4 k \frac{k \cdot q}{k^2 G(x, \log k^2) (k + q)^2} - \dots \Big|_{q^2 = \mu^2}$$

where $L = \log(q^2/\mu^2)$.

The differential equation is

$$\gamma_1(x) = x - \gamma_1(x)(1 - 2x\partial_x)\gamma_1(x)$$

Solving (presented in the beautiful form given by Broadhurst)

$$\sqrt{\frac{x}{2\pi}} = \exp \left(\left(\frac{(\gamma_1 + 2)^2}{\sqrt{2x}} \right)^2 \right) \operatorname{erfc} \left(\frac{(\gamma_1 + 2)^2}{\sqrt{2x}} \right)$$

A variant

Take $s = 2$ and

$$F(\rho) = \frac{-1}{\rho(1-\rho)(2-\rho)(3-\rho)},$$

$$\rho F(\rho) = -\frac{1}{6} + \rho^2 F(\rho) - \frac{11}{6} \rho^3 F(\rho) + \frac{1}{6} \rho^4 F(\rho)$$

This gives that

$$\begin{aligned} \gamma_1 &= -x(1 - \gamma \cdot \partial_{-\rho})^{-1} \rho F(\rho)|_{\rho=0} \\ &= -x(1 - \gamma \cdot \partial_{-\rho})^{-1} \left(-\frac{1}{6} + \rho^2 F(\rho) - \frac{11}{6} \rho^3 F(\rho) + \frac{1}{6} \rho^4 F(\rho) \right) \Big|_{\rho=0} \\ &= \frac{x}{6} - 2\gamma_2 - 11\gamma_3 - 4\gamma_4 \end{aligned}$$

Concluding the variant

So

$$\gamma_1 = \frac{x}{6} - 2\gamma_2 - 11\gamma_3 - 4\gamma_4.$$

But we still have

$$\gamma_k = \frac{1}{k}\gamma_1(x)(1 - 2x\partial_x)\gamma_{k-1}(x),$$

So we get a fourth order differential equation for γ_1 which contains no infinite series and for which we completely understand the signs of the coefficients.

Bonus slides – Growth estimates

View

$$\gamma_1(x) = P(x) - \gamma_1(x)(\text{sign}(s) - |s|x\partial_x)\gamma_1(x)$$

as a recursive equation. At the level of coefficients

$$\begin{aligned} \gamma_1(x) &= \sum \gamma_{1,n} x^n \\ \text{so } \gamma_{1,n} &= p^{(n)} - \sum_{\ell=1}^{n-1} \gamma_{1,n-\ell} (\text{sgn}(s) - |s|\ell) \gamma_{1,\ell} \\ &= p^{(n)} - \sum_{\ell=1}^{n-1} \left(\text{sgn}(s) - \frac{|s|\ell}{2} \right) \gamma_{1,\ell} \gamma_{1,n-\ell} \end{aligned}$$

Rewrite for $a(n)$

Assume $\gamma_{1,1} \neq 0$ and $f(x) = \sum \frac{p(n)}{n!} x^n$ has radius of convergence $\rho > 0$.

Let $a(n) = \frac{\gamma_{1,n}}{n!}$. The recursion becomes

$$a_n = \frac{p(n)}{n!} + \left(\frac{|s|^n}{2} - \operatorname{sgn}(s) \right) \sum_{l=1}^{n-1} a_l a_{n-l} \binom{n}{l}^{-1}$$

$$=$$

How bad is the growth of γ_1 ?

Idea:

$$a(n) \text{ is approximately } \frac{p(n)}{n!} + |s|a_1a_{n-1}$$

giving a radius of $\min \left\{ \rho, \frac{1}{sa_1} \right\}$ for $\sum a_n x^n$. For nonnegative series implement the idea by bounding on each side.

Easy direction:

$$a_n \geq \frac{p(n)}{n!} + |s| \frac{n-2}{n} a_1 a_{n-1}$$

Messy direction: for any $\epsilon > 0$ there is an $N > 0$ such that for $n > N$

$$a_n \leq \frac{p(n)}{n!} + |s|a_1a_{n-1} + \epsilon \sum_{j=1}^{n-1} a_j a_{n-j}$$