

# Recent progress on an arithmetic graph invariant with applications in quantum field theory.

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# Numbers in Feynman integrals.

As many of you know, interesting numbers show up in perturbative quantum field theory calculations.

These *interesting numbers* include

multiple zeta values  
other values of multiple polylogarithms  
elliptic generalizations and onwards

# What are these calculations?

- In perturbative quantum field theory you calculate physical things using series indexed by Feynman diagrams – certain graphs describing particle interactions.
- Each Feynman diagram contributes an integral where the integrand is built out of pieces corresponding to edges and vertices in the graph.
- Or you can try to be more clever and index the sum in other ways.

# Where do the interesting numbers appear?

These numbers appear almost no matter what you do.

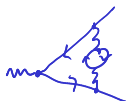
- These numbers appear in Feynman integrals for different graphs.
- They appear in essentially any interesting choice of quantum field theory

two examples

$\phi^4$



QED



- They appear in the sum (not just in the individual graphs).
- There are patterns to how they appear, but it is still mysterious.

# Why do the numbers appear?

Why indeed?

MZVs should come  
from mixed Tate motives  
etc.



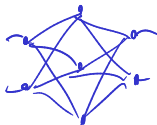
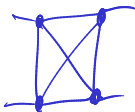
There must be some good mathematical reason.

That would be interesting mathematics and it would explain something about Feynman integrals and hence let us do more physics too.

# Keep it simple

One example of a quantum field theory is  $\phi^4$

one edge type —  
one vertex 



These will be our graphs today.

# The Kirchhoff polynomial

Let  $K$  be a connected 4-regular graph

Let  $G = K - v$ . These are connected  $\phi^4$  graphs with 4 external edges.

Assign a variable for each edge  $a_e$  for edge  $e$

Define

$$\Psi_G = \sum_{\substack{T \\ \text{spanning tree} \\ \text{of } G}} \prod_{e \notin T} a_e$$

Eg:

$$\Psi = cd + ad + bd + bc + ac$$



# The Feynman period

Rather than consider the full Feynman integral and what numbers and functions of the parameters show up, we'll just think about:

$$P_G = \int_{a_i \geq 0} \frac{da_1 \cdots da_{n-1}}{\psi_G^2} \Big|_{a_n=1}$$

Eg:



$$\psi = a + b$$

$$\int_{a \geq 0} \frac{1}{(a+b)^2}$$

## Period – geometry – arithmetic

to understand  $P_G$  without doing the integral

I want to understand the geometry

$$\text{of } \psi_G = 0$$

Another direction into this geometry, count points  
on  $\psi_G$  over finite fields

# The $c_2$ invariant

For  $f \in \mathbb{Z}[x_1, \dots, x_n]$  define  $[f]_q$  to be the number of  $\mathbb{F}_q$ -rational points on the variety  $f = 0$ .

Define

$$c_2^{(p)}(G) = \frac{[\psi_G]_p}{p^2} \mod p$$

# Arithmetic structure

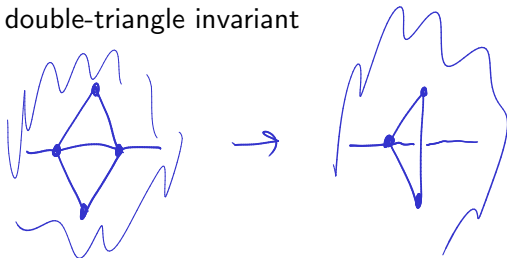
- If  $c_2^{(p)}(G)$  is independent of  $p$  then  $P_G$  should be MZV.
- If  $c_2^{(p)}(G) = 0$  then  $P_G$  should have less than maximal transcendental weight.
- If  $c_2^{(p)}(G)$  is constant in some field extension then  $P_G$  should be a multiple polylogarithm evaluated at the roots.
- Some  $c_2^{(p)}(G)$  are proven to be coefficient sequences of modular forms.
- In this case  $P_G$  should be more exotic.

# Known graph-related properties

the 4 regular graph



- If  $K$  has a 3-separation then  $c_2^{(p)}(G) = 0$ .
- If  $K$  has an internal 4-edge-cut then  $c_2^{(p)}(G) = 0$ .
- If  $G$  has vertex width 3 then  $c_2^{(p)}$  is a constant.
- $c_2$  is double-triangle invariant



# Known and conjectured symmetries

$p_G$

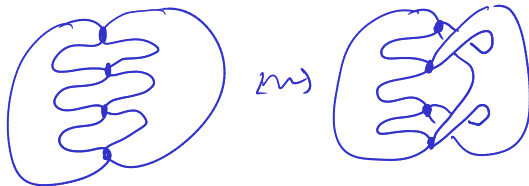
The period is proven to be invariant under

- Completion/decompletion

$$G = K - v \quad \leftarrow \text{what } v \text{ doesn't matter}$$

- Planar duality for  $G$
- Schnetz twist

on  $K$



The  $c_2$  invariant should have these symmetries as well.



# A result

Brown and Schnetz conjecture that for all  $p$ , 4-regular  $K$ ,  
 $v_1, v_2 \in V(K)$

$$c_2^{(p)}(K - v_1) = c_2^{(p)}(K - v_2)$$

I prove that if  $K$  has an odd number of vertices,  $v_1, v_2 \in V(K)$ ,  
then

$$c_2^{(2)}(K - v_1) = c_2^{(2)}(K - v_2)$$

# Expanded Laplacian and more polynomials

Let

$$M_G = \begin{bmatrix} \Lambda & E^T \\ -E & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} \text{edges} \\ \text{vertices} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

where  $\Lambda = \text{diag}(a_1, a_2, \dots, a_n)$  and  $E$  is the signed incidence matrix with one row removed.

Then as another way to view the matrix tree theorem we have

$$\Psi_G = \det M_G$$

We also care about minors

$$\Psi_{G,K}^{I,J} = \det M_G(I, J) \Big|_{a_e=0, e \in K}$$

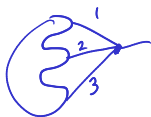
*INT is deletion*  
*remove rows indexed by I*  
*remove columns indexed by J*  
*contraction*

# One known result we need

## Proposition (Brown and Schnetz)

$$c_2^{(p)}(G) = [\Psi_{G,3}^{1,2} \Psi_G^{13,23}]_p \pmod p$$

In particular if edges 1, 2, 3 meet at a 3-valent vertex:



$$\Psi_3^{1,2} = \text{edges 1, 2 meet}$$

$$\text{and } \text{graph with loop} = \text{graph with loop}$$

$$\Psi^{13,23} = \text{graph with loop} = \Psi$$

# Another known result we need

## Proposition (Corollary of Chevalley-Warning)

*If  $f$  has total degree  $n$  in  $x_1, x_2, \dots, x_n$  then*

$$[f]_p = \text{coefficient of } x_1^{p-1} \cdots x_n^{p-1} \text{ in } f^{p-1} \pmod{p}$$

# Reduction to counting certain edge bipartitions

Apply these to our situation.

$$c_2^{(2)}(G) = [\psi_{G,3}^{1,2} \psi_G^{13,23}]_2 \pmod{2}$$

= coefficient of  $x_1 \cdots x_n$  in  ~~$\psi_{G,3}^{1,2} \psi_G^{13,23}$~~   $\pmod{2}$

let  $G =$  

$H =$  

= number of ways of partitioning  
the edges of  $H$  so that one  
part is a spanning tree and  
the other is a spanning forest corr. to



# Proof sketch



similar but easier if  $v_1, v_2$  have  
common neighbors

$$\begin{aligned}
 & \left( \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \end{array} + \dots + \begin{array}{c} \text{diagram 4} \end{array} \right) + \begin{array}{c} \text{diagram 5} \end{array} + \begin{array}{c} \text{diagram 6} \end{array} \\
 & + \left( \begin{array}{c} \text{diagram 7} \\ \text{diagram 8} \end{array} + \dots \right) \begin{array}{c} \text{diagram 9} \end{array} + \begin{array}{c} \text{diagram 10} \end{array} \pmod{2}
 \end{aligned}$$

$$= \text{[diagram of 6 terms]} \pmod 2$$

+ swap top and bottom

couldn't find something like  
 instead can swap around cycles

build an auxiliary graph where the vertices are the  
 things I want to count and joined if you can swap between them

find a fixed  
 point free involution  
 so evenly many

↑  
This auxiliary graph has all vertices  
of odd deg (here even # of them)  
when  $K$  has an odd number of vertices