Equivalences of Wilson loop diagrams

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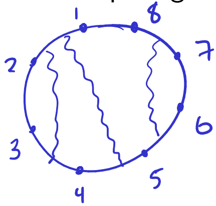
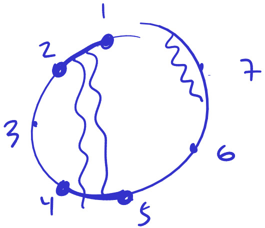
Joint work with Susama Agarwala and Zee Fryer.
arXiv:1908.10919 and arXiv:1910.12158

A different bit of combinatorics in QFT than my usual story

A Wilson loop diagram:

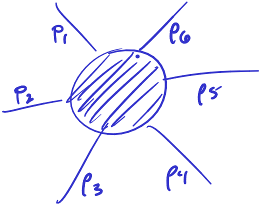




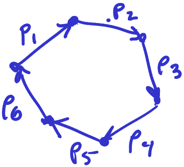
The diagram is admissible if there is no crossing and it is not too dense; no set of propagators P is supported on less than $|P| + 3$, vertices and not even equality for the whole diagram.

Idea

The idea of the Wilson loop is essentially duality.



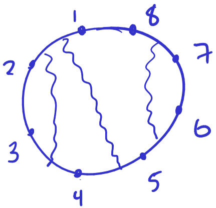
$$p_1+p_2+p_3+p_4+p_5+p_6=0$$



The propagators encode helicity violation.

Wilson loop diagram to matrix. . .

You go from a Wilson loop diagram to a matrix as follows:

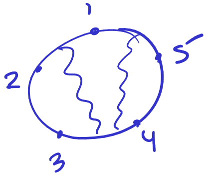


$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \end{array}$$
$$\begin{bmatrix} a & b & 0 & c & d & 0 & 0 & 0 \\ e & f & g & h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & j & k & l \end{bmatrix}$$

. . . to positroid

The matrix it represents is a positroid, that is for appropriate choices of the variables, all the maximal minors are nonnegative.

Eg



$$\begin{bmatrix} a & b & c & d & 0 \\ 0 & e & f & g & h \end{bmatrix}$$

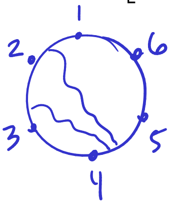
$$\left.\begin{array}{l} bf > ce \\ cg > df \end{array}\right\} \quad \frac{b}{e} > \frac{c}{f} > \frac{d}{g}$$

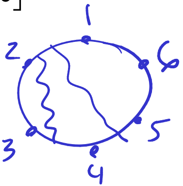
and $a,b,c,d,e,f,g,h > 0$

But they are not always different positroids

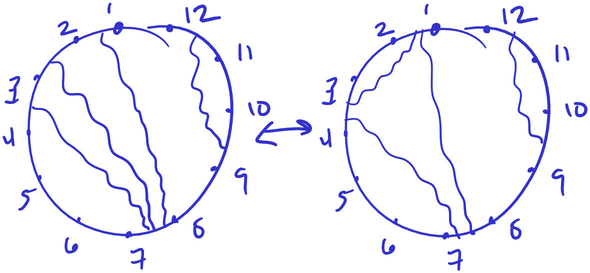
Eg

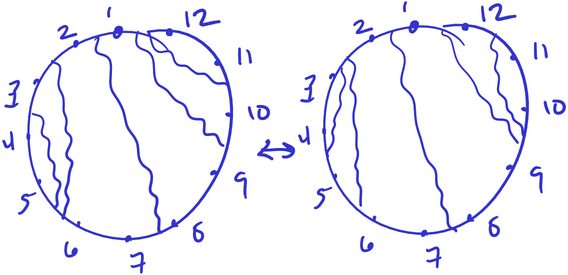
$$\begin{bmatrix} a & b & 0 & c & d & 0 \\ 0 & e & f & g & h & 0 \end{bmatrix} \qquad \begin{bmatrix} a & b & 0 & c & d & 0 \\ e & f & g & h & 0 & 0 \end{bmatrix}$$





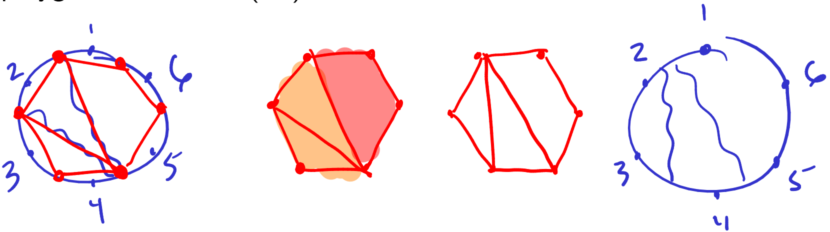
Some bigger examples





Triangulations

The key is triangulations. Convert a Wilson loop diagram W to a polygon dissection $\tau(W)$:

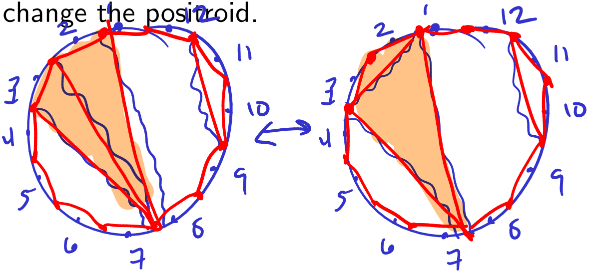


Result

Theorem

Two admissible Wilson loop diagrams W and W' define the same positroid if and only if $\tau(W)$ and $\tau(W')$ differ by retriangulations.

Idea of proof: an exact subdiagram is one which is critical for the density requirement. Replacing one exact subdiagram with another is retriangulating. Replacing one exact subdiagram with another does not change the positroid.



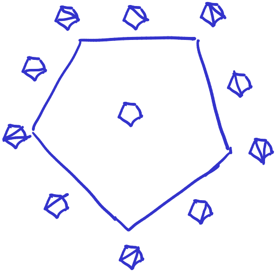
Counting

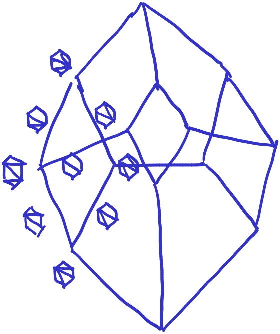
Triangulations are counted by Catalan, so the number of admissible Wilson loop diagrams giving the same positroid where the sizes of the nontrivial maximal exact subdiagram are $n_1, n_2, \ldots, n_j$ is

$$\prod_{i=1}^{j} \frac{1}{n_i - 1} \binom{2(n_i - 2)}{n_i - 2}$$

Associahedra

Associahedra can also be defined by polygon dissections. Eg:



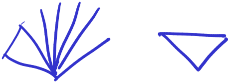


Realized

Let $x_1, \ldots, x_n$ be the corners of a convex n-gon in $\mathbb{R}^2$ and let T be the set of triangulations of this n-gon.

For each $t \in T$, define s_t to be the point in $\mathbb{R}^n$ with ith coordinate the sum of the areas of all triangles of t incident to x_i.

Let A_n be the convex hull of the s_t.

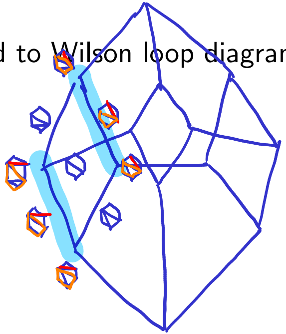


A_n is a realization of the $n-1$ associahedron.

Parallelism

Parallel faces correspond to Wilson loop diagrams giving the same positroid.

Non-parallel faces correspond to Wilson loop diagrams giving different positroids.



This is the retriangulations again. The second direction takes some care with the bigger degree faces.

What is the dimension of the positroid cell?

Answer, 3 times the number of propagators. Can we understand this explicitly?

⇝ 2017 Mainz
could I prove $\dim = 3|P|$

⇝ induction!

⇝ geometric proof Agarwala + Marcott

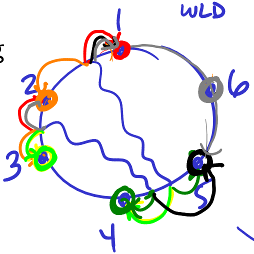
⇝ later we ultimately wrote the 2017 proof
and other things
⇝ see coverpage.

Move to the Le diagram

There are many nice combinatorial objects in bijection with positroids. The one we want is the Le diagram.

WLD

Eg



Grassm. necklace

12, 23, 34, 45, 51, 12

Le diagram.

	6	5	4	3	
1	0	+	+	+	1
2	0	+	+	+	2
	6	5	4	3	

The dimension is the number of plusses

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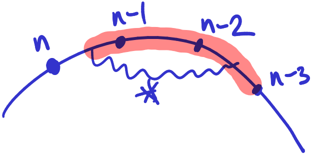
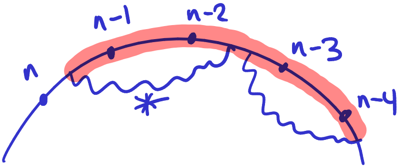
We can make use of this . . . via a very annoying induction.

Step 1 dihedral trans preserve dim.
remove vertices that don't support any propagators

so WLOG



or

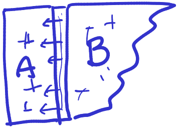


Sketch

② induction → remove *

get

without *



has *